Introduction to Discrete Mathematics

Pascal's triangle has fascinated mathematicians for centuries. Though simple to generate, it is a rich field for demonstrating the basics of sequences and combinatorics as well as a perfect introduction to proofs in discrete mathematics.

1 Pascal's Magic Triangle

$$
\begin{array}{l ccccccccccccccccc}
n=0 & & & & & & & & & 1 & & & & & & & & \\
n=1 & & & & & & & & 1 & & 1 & & & & & & & \\
n=2 & & & & & & & 1 & & 2 & & 1 & & & & & & \\
n=3 & & & & & & 1 & & 3 & & 3 & & 1 & & & & & \\
n=4 & & & & & 1 & & 4 & & 6 & & 4 & & 1 & & & & \\
n=5 & & & & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & & & \\
n=6 & & & 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 & & \\
n=7 & & 1 & & 7 & & 21 & & 35 & & 35 & & 21 & & 7 & & 1 & \\
n=8 & 1 & & 8 & & 28 & & 56 & & 70 & & 56 & & 28 & & 8 & & 1 \\
\hline
 & 0 & & 1 & & 2 & & 3 & & 4 & & 5 & & 6 & & 7 & & 8
\end{array}
$$

The figure above is Pascal's Triangle taken to nine rows, numbered as $n = 0$ to 8. With a first row of just 1, every next row number to left and right in previous row. (0 outside the triangle). What patterns can we find. Factors of x and y with the x power decreasing by 1, and the corresponding y power increasing by 1 staring from y^0, or 1.

2 Binomial Expansion

When expanding $(x + y)^n$ we must avoid the 'Freshman Mistake' of taking the individual terms to the n, such as $\cancel{x^n + y^n}$ which of course is **ABSOLUTELY INCORRECT!** Distributing, or as some call it FOILing. FOILing of course generates 4 terms which makes for a nice abbreviation, but what we are really doing is distributing. And for powers greater than 2 there are of course more than 4 terms, so FOILing is a very limiting term to use. Just make sure every term in one factor gets a turn to multiply every other term in the other factor exactly once. We know from algebra I that the correct expansion of $(x + y)^2 = (x + y)(x + y)$ is: $x^2 + 2xy + y^2$. We see that the pattern from the third row ($n = 2$) of the triangle gives the correct coefficients, and the power rule for the x and y factors hold as well. How about for $(x + y)^3$? This can be written as $(x + y)(x + y)^2$, and we already did the expansion for $(x + y)^2$, so we have $(x+y)^3 = (x+y)(x+y)^2 = (x+y)(x^2+2xy+y^2) = x^3+2x^2y+xy^2+yx^2+2xy^2+y^3$. Combining like terms and writing terms starting from highest power of x we have: $x^3+3x^2y+3xy^2+y^3$. Once again this fits the corresponding row of the triangle, for n=3 (fourth row). Try expanding $(x + y)^4$, using the fact that $(x + y)^4 = (x + y)^2(x + y)^2$. Then use the triangle pattern to check your work. The answer is in the section at the end of this packet.

3 Binomial Theorem

Choosing k items from a total of n, without regard to ordering is denoted $\binom{n}{k}$ and is defined to be: $\frac{n!}{k!(n-k)!}$, where the exclamation point (bang) denotes factorial, the pattern for multiplying from the given argument down to 1, viz.: $5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$

1	$n = 0, \sum = 1$
1 1	$n = 1, \sum = 2$
1 2 1	$n = 2, \sum = 4$
1 3 3 1	$n = 3, \sum = 8$
1 4 6 4 1	$n = 4, \sum = 16$
1 5 10 10 5 1	$n = 5, \sum = 32$
1 6 15 20 15 6 1	$n = 6, \sum = 64$
1 7 21 35 35 21 7 1	$n = 7, \sum = 128$
1 8 28 56 70 56 28 8 1	$n = 8, \sum = 256$

If we look at the sums of the rows we see a pattern. Doubling every row, powers of 2 (including $1 = 2^0$). Why is that? The definition of binomial is there are only 2 possible values for each factor, there or not there, True of False, etc. By the counting principle, if choices are independent we multiply number of choices. For example for 5 lights that can be either on or off we would have $\underline{L1}$ $\underline{L2}$ $\underline{L3}$ $\underline{L4}$ $\underline{L5}$. If each light can only be either on or off, and the state of each light is independent of all others, then by the counting principle we would count our choices via multiplication, viz: $\underline{2} \cdot \underline{2} \cdot \underline{2} \cdot \underline{2} \cdot \underline{2} = 2^5 = 32$

Let us use the example of asking how many ways a specific number of heads can come up (without regard to order) when 6 coins are flipped (either simultaneously or sequentially so long as one flip does not affect another). It is easy to see that there is only 1 way to get all 6 heads, and one way to get no heads at all. It is also easy to see that there are 6 ways exactly one head can occur, as well as 6 ways exactly 5 heads (1 tail) can occur. But what about other counts of heads results? We could list each possible outcome and tally any entry that contains 3 heads, but for six coins this will be difficult, and situations where there are more than 6 coins involved very quickly becomes quite impractical. We can once again use the counting principle. Because we do not count different orderings of the three heads we must divide out duplicate entries $\rightarrow$ entries with 3 heads in different orders. In other words, a combination not a permutation. $\frac{6!}{3!3!}$, denoted as $\binom{6}{3}$and equal to $\frac{720}{6 \cdot 6} = 20$

4 Sequences & Series

4.1 Sequences

4.1.1 Arithmetic

XXX

4.1.2 Others

4.2 Series

4.2.1 Arithmetic

Closed form

4.2.2 Geometric

Closed form

4.2.3 Others

5 Summation

5.1 Notation and Indexing

5.1.1 Notation

$\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} = 2$. Due to the vertical span that can interfere with line spacing when printed inline with text, (as you can see here) many documents now use this style for inline summation formulas: $\sum_{n=1}^{\infty} \frac{1}{n^2} = 2$.
Properties, double summation, etc.

5.1.2 Index Variables

XXX

5.2 Properties and Algebraic Manipulation

$$\sum_{i=0}^{n} af(i) = f(n) \implies a\sum_{i=0}^{n} f(i) = n$$
$$\sum_{i=0}^{n} f(i) = f(n) \implies f(0) + \sum_{i=1}^{n} = f(0) + f(n)$$
$$\sum_{i=0}^{n-1} f(i) = f(n) \implies \sum_{i=1}^{n} f(i-1) = f(n)$$

Check for understanding. Fill in the boxes:

$$\sum_{i=1}^{n+1} f(i-1) = f(n) \implies \sum_{i=0}^{\square} f(\square) = f(n)$$

5.2.1 Linearity

XXX

5.2.2 Associative, Commutative

XXX

6 Algebraic Proof of Pascal's Identity

$$\binom{n-1}{k-1} + \binom{n-1}{k} = \frac{(n-1)!}{k!(n-1-k)!} + \frac{(n-1)!}{(k-1)!(n-k)!} \tag{1}$$

$$= (n-1)!\Big[\frac{n-k}{k!(n-k)!} + \frac{k}{k!(n-k)!}\Big] \tag{2}$$

$$= (n-1)!\frac{n}{k!(n-k)!} \tag{3}$$

$$= \frac{n!}{k!(n-k)!} = \binom{n}{k} \tag{4}$$

7 Proof by Induction of Binomial Theorem

7.1 Proof Outline

Induction yields another proof of the binomial theorem. When $n = 0$, both sides equal 1, since $x^0 = 1$ and $\binom{0}{0} = 1$. Now suppose that the equality holds for a given n; we will prove it for $n+1$. For $j, k \geq 0$, let $f(x, y)_{j,k}$ denote the coefficient of x^j, y^k in the polynomial $f(x, y)$. By the inductive hypothesis,$(x + y)^n$ is a polynomial in x and y such that $[(x+y)^n]_{j,k}$ is $\binom{n}{k}$ if $j+k = n$, and 0 otherwise. The identity $[(x + y)^{n+1}]_{j,k} = [(x + y)^n]_{j-1,k} + [(x + y)^n]_{j,k-1}$, shows that $(x + y)^{n+1}$ is also a polynomial in x and y,and since if $j + k = n + 1$ then $(j - 1) + k = n$ and $j + (k - 1) = n$. Now, the right hand side is $\binom{n}{k-1} = \binom{n+1}{k}$ by *Pascal's identity* (proven in **Section 5** above). On the other hand, if $j + k \neq n + 1$, then $j - 1 + k \neq n$ and $j + (k - 1) \neq n$, so we get $0 = 0 + 0$. Thus $(x + y)^{n+1} = \sum\limits_{k=0}^{n+1} \binom{n+1}{k} x^{n+1-k} y^k$ which is the inductive hypothesis with $n + 1$ substituted for n and so completes the inductive proof for all $n \in \mathbb{N}^*$

7.2 In Depth Proof

In order to show that the binomial theorem is true for $n \in \mathbb{N}^*$, we will first prove that it is true for $n = 0$, then for $n \in \mathbb{N}^*$ by induction.

Proof for n=0

We have: $\sum\limits_{k=0}^{n} \binom{n}{k} x^{n-k} y^k = \binom{0}{0} x^0 y^0 = 1$
We then deduce that the binomial theorem is true for $n = 0$.

INDUCTION

We know that for $n = 0$, we have: $(x+y)^n = \sum\limits_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$, so for $n+1$, we have:

$(x+y)^{n+1}$

$= (x+y)(x+y)^n$

$= x(x+y)^n + y(x+y)^n$

$= x\left(\sum\limits_{k=0}^{n} \binom{n}{k} x^{n-k} y^k\right) + y\left(\sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k\right)$

$= \sum\limits_{k=0}^{n} \binom{n}{k} x^{n-k+1} y^k + \sum\limits_{k=0}^{n} \binom{n}{k} x^{n-k} y^{k+1}$

$= \sum\limits_{k=0}^{n} \binom{n}{k} x^{n-k+1} y^k + \sum\limits_{k=1}^{n+1} \binom{n}{k-1} x^{n-k+1} y^k$

$= \binom{n}{0} x^{n+1} y^0 + \left[\sum\limits_{k=1}^{n} \binom{n}{k} x^{n-k+1} y^k + \sum\limits_{k=1}^{n} \binom{n}{k-1} x^{n-k+1} y^k\right] + \binom{n}{n} x^0 y^{n+1}$

$= x^{n+1} + \left[\sum\limits_{k=1}^{n} \binom{n}{k} x^{n-k+1} y^k + \sum\limits_{k=1}^{n} \binom{n}{k-1} x^{n-k+1} y^k\right] + y^{n+1}$

$= x^{n+1} + \left[\sum\limits_{k=1}^{n} \left(\binom{n}{k} + \binom{n}{k-1}\right) x^{n-k+1} y^k\right] + y^{n+1}$

$= x^{n+1} + \left[\sum\limits_{k=1}^{n} \binom{n+1}{k} x^{n-k+1} y^k\right] + y^{n+1}$

Because we know that the binomial theorem is true for $n = 0$ and for $n = n+1$, we deduce by induction that it is true for $n \in \mathbb{N}^*$.

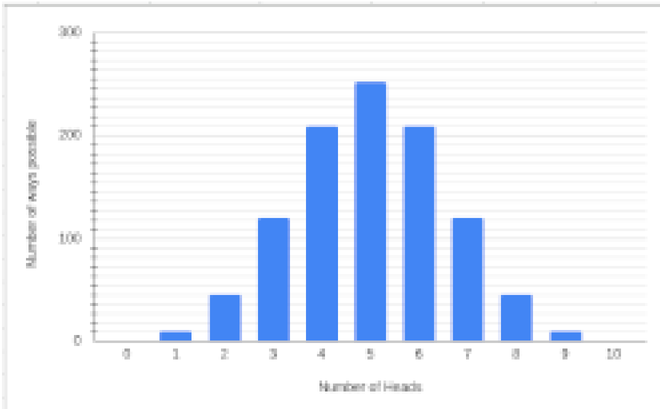
Thinking Questions

Why must every row of the triangle sum to 2^n (where n is the row number)?

Answers to select exercises

$(x+y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4.$

8 Binomial Distribution



The graph shown above is a bar chart of the possible number of ways to get various numbers of heads (from 0 to 10) from flipping 10 coins. It does not matter if it is done sequentially or all at once as long as the crucial condition of independence is met, i.e. no one coin flip affecting another. For many more coins this starts to approach what is called the normal distribution, and when there are a great many coin flips (in the thousands), the actual numbers will be almost identical to these theoretical values. It is why the normal distribution occurs so much in the real world, where occurrence of a great many independent events are tallied and charted. The reason the real numbers will always approach these theoretical numbers from the binomial theorem (or the rows of Pascal's Triangle) is called the central limit theorem.

Legal Matter

$$
\begin{array}{ccccccccccccccccccl}
 & & & & & & & & 1 & & & & & & & & & & n=0, \sum = 1 \\
 & & & & & & & 1 & & 1 & & & & & & & & & n=1, \sum = 2 \\
 & & & & & & 1 & & 2 & & 1 & & & & & & & & n=2, \sum = 4 \\
 & & & & & 1 & & 3 & & 3 & & 1 & & & & & & & n=3, \sum = 8 \\
 & & & & 1 & & 4 & & 6 & & 4 & & 1 & & & & & & n=4, \sum = 16 \\
 & & & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & & & & & n=5, \sum = 32 \\
 & & 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 & & & & n=6, \sum = 64 \\
 & 1 & & 7 & & 21 & & 35 & & 35 & & 21 & & 7 & & 1 & & & n=7, \sum = 128 \\
1 & & 8 & & 28 & & 56 & & 70 & & 56 & & 28 & & 8 & & 1 & & n=8, \sum = 256
\end{array}
$$

$\bar{[}.]$